Succinct Preferential-Attachment Graphs

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- We create a vertex v_1.
- At each step t from 2 to n, we create one vertex v_t and direct M edges from v_t to $v_{t'}$, where $t' < t$.
- The probability of $v_{t'}$ to be selected as an out-neighbour of v_t is proportional to its degree right before v_t was created.

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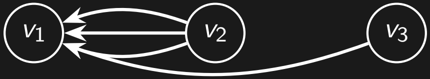

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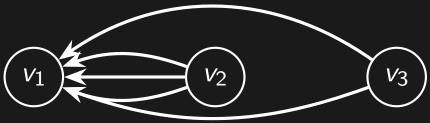

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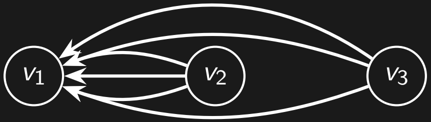

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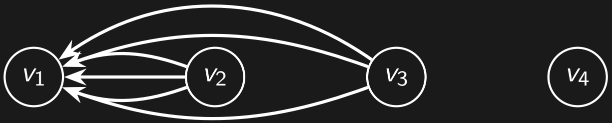

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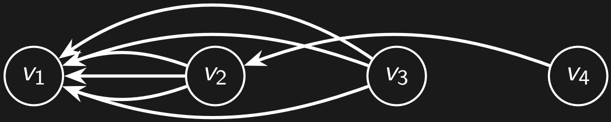

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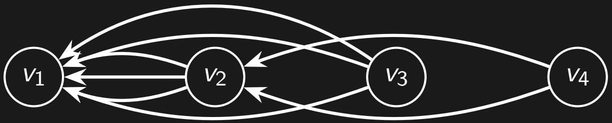

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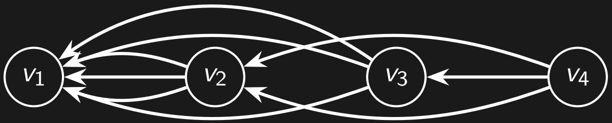

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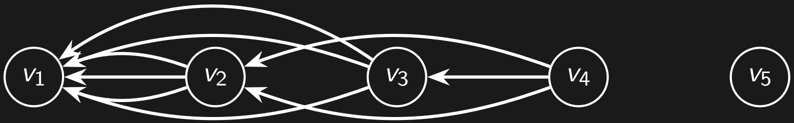

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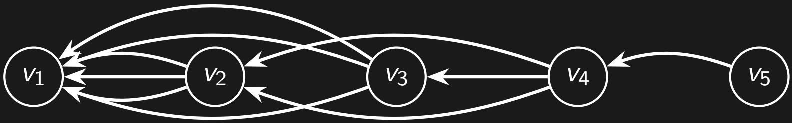

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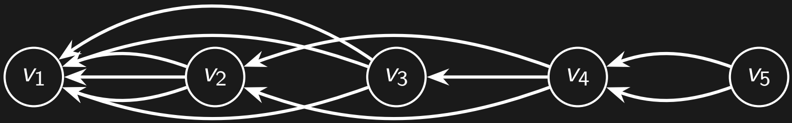

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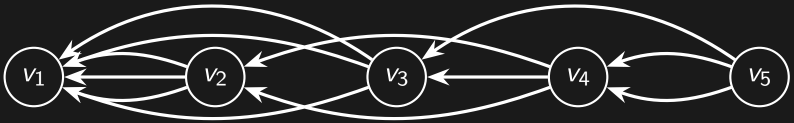

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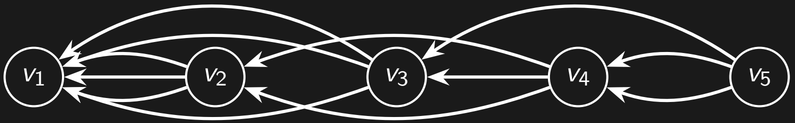

1 $\quad$ 1 $\quad$ $1 \cdot (3/6)^3$ $\quad$ $3 \cdot (3/12)^3$ $\quad$ $3 \cdot (3/18)^2 \cdot (4/18)$

Thus, the probability of the graph G_5 being generated is $\mathbb{P}[G_5] = 1/9216$.

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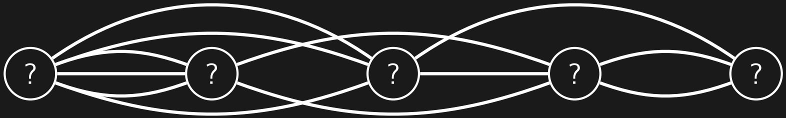

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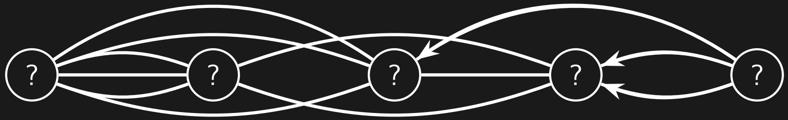

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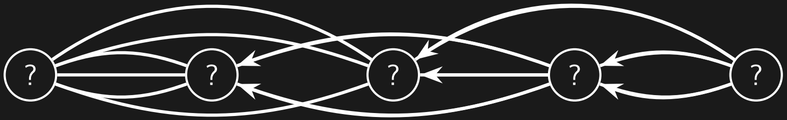

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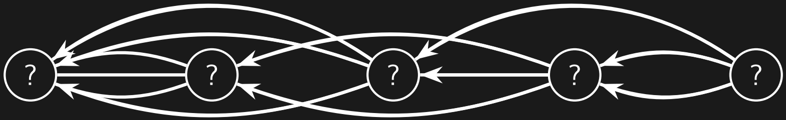

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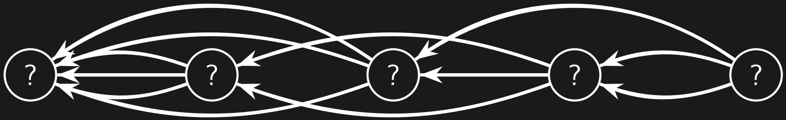

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Let G be a directed graph. An *adjacency string* A of G is a string whose alphabet $\Sigma = V(G)$, and $A = N^+(v_1)N^+(v_2)\dots N^+(v_n)$ where $N^+(v)$ denotes a string concatenating the out-neighbours of v in some arbitrary order.

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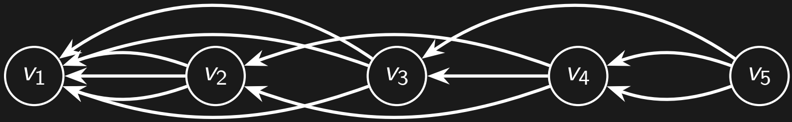

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Theorem

Given an n-vertex directed graph G generated by the BA model with probability p, $\lg(1/p) = H_0(A) \pm O(nM \lg M)$.

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- **One solution:** wavelet trees!

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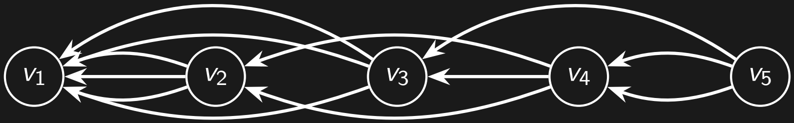
A is of size nM. So we can compress A in $H_0(A) + o(nM)$ bits while supporting the above operations in $O(\lg n)$ time (here, $\sigma = n$).

Navigational operations on G using A (1/3)



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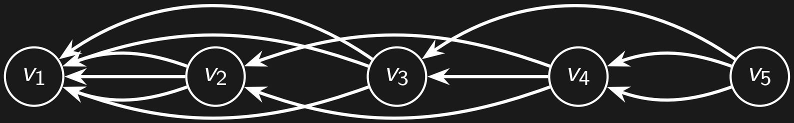

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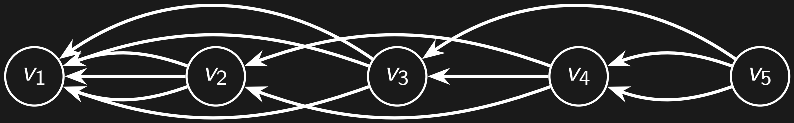

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Solution: Compute $A[M(i-1)+j]$ using a $O(\lg n)$ access query on A.

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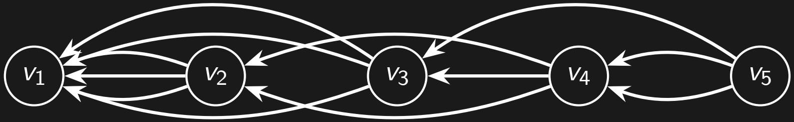

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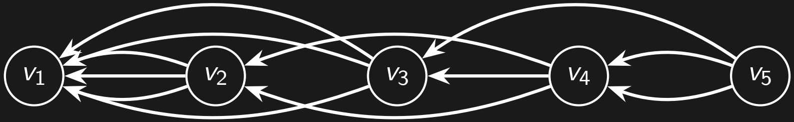

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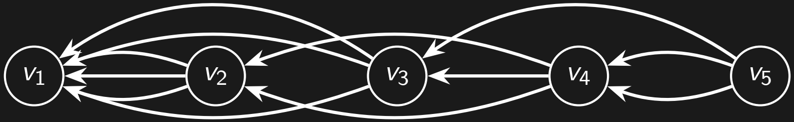

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Solution: Count how many times v_i occurs in A. A simple `rank` query suffices; $O(\lg n)$.

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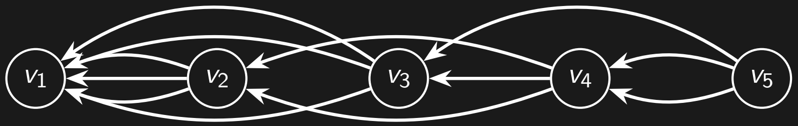

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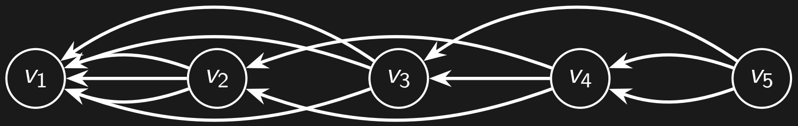

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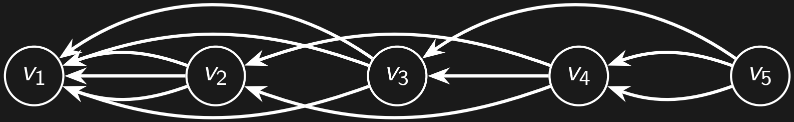

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Solution: Find the position x in A where the jth occurrence of v_i is. The answer is $\lceil x/M \rceil$. The first part involves a `select` query; $O(\lg n)$.

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- This scheme can be extended to general graphs (i.e. non-M-regular graphs) by using an additional vector of bits that denote the sizes of the out-neighbourhoods in A.

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Cheers.